Exploring deltahedra in the primary classroom

Tandi Clausen-May shares some interesting discoveries that children have made exploring shapes made with triangles in her mathematics workshops.

As the guiding principles of the ATM state, “The ability to operate mathematically is an aspect of human functioning which is as universal as language itself.” So operating mathematically is at the heart of the mathematics enrichment workshops that I run in primary schools. It is not the specific content of the activities that matters: it is the process of exploring new ideas and discovering things that we did not know. And, just occasionally, quite unexpectedly, we discover something that perhaps not many other people know yet either. That can be incredibly exciting, for all of us.

Using squares to make a net that will fold up into a cube is a standard classroom activity for primary school children. Triangle-faced 3-D shapes, or ‘deltahedra’, on the other hand, are less familiar, so they can offer an exciting and engaging new challenge. As we explore the deltahedra and their nets with ATM MATs and *Polydron* equilateral triangles we work together as a community of mathematicians, operating mathematically, to move our knowledge and understanding forward. This has sometimes led children to make interesting discoveries.

Ribbon nets

A few years ago, I was taken by surprise when two boys, Josh and Len, said that they had found a net made of eight triangles that would fold up into two different octahedra (Clausen-May 2016). They claimed that their net could be folded into the *Regular* octahedron, but that it would also fold into the other delta-octahedron, the *Boat* (Figure 1).

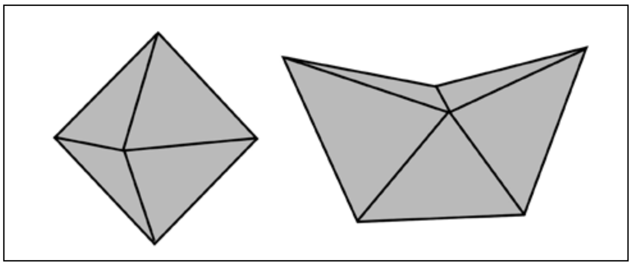

Figure 1: The *Regular* and the *Boat* Octahedra.

I was dubious. How could one net make two different shapes? But it was true, as I eventually realised. Since then, I have been more careful what I say when a child discovers something I did not know.

So now I encourage everyone in the workshops to look for the nets of both the *Regular* and the *Boat* octahedra. We display them separately, stopping to think about it if we find the same net in both categories. The *Regular* octahedron has only eleven nets, and images of these are readily available in books or on the internet. Seven of the *Regular* octahedron nets are doubles like the one that Josh and Len found: they can fold up into either of the two delta-octahedra (Figure 2).

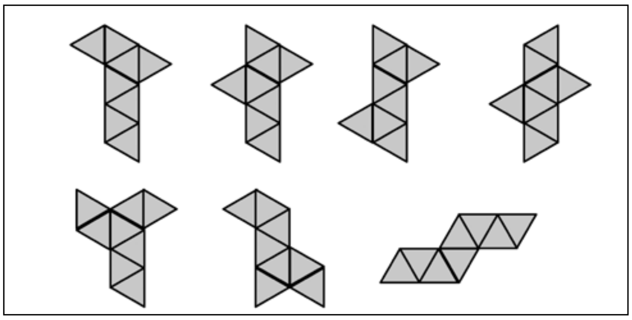

Figure 2: The seven nets that will form both the *Regular* and the *Boat* octahedra.

But there are many more nets of the *Boat*, and I have been unable to track down any information about these. Figure 3 shows a ‘work in progress’ display from one of our workshops, where the children had found 14 correct nets of the *Boat* (plus a couple of mistakes and a repeat), but there may well be more.

Figure 3: Looking for octahedra nets.

So, the nets of the *Boat* are many and varied, but nevertheless I had my doubts when 10-year-old Evie cut out a net composed of a straight-line ribbon of eight triangles and stuck it up on the display board under *Boat*. "Are you sure?" I asked. "Will that net really fold up into the *Boat*?" Evie nodded confidently, then dashed back to her table to look for another one. I did not challenge her any further this time (I had learnt my lesson from Josh and Len), but I made a note to myself to look at this strange net later.

Evie was right, of course. When I got home, I made the 8-triangle ribbon net. It folded up into the *Boat* with no trouble (see Figure 4).

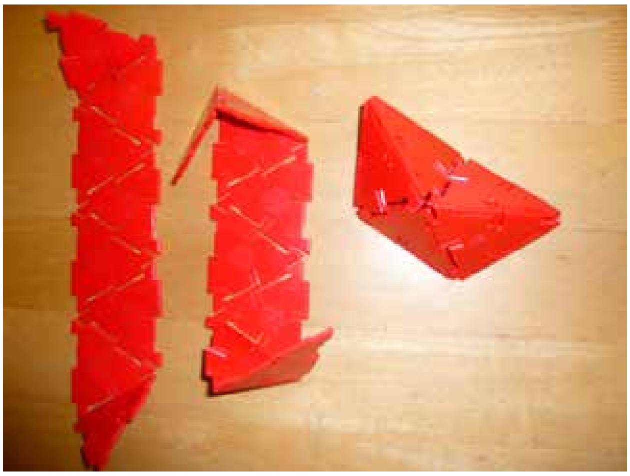

Figure 4: Folding an 8-triangle ribbon net into a *Boat* octahedron.

This discovery led me to think about other ribbon nets. A ribbon of four equilateral triangles is one of the two nets of a regular tetrahedron, while a ribbon of six is one of the eight nets of a triangular bipyramid (Figure 5). The 8-triangle ribbon, as we have seen, is one of many (I don't know how many) that will fold up into the *Boat*. So, what can be done with ten triangles?

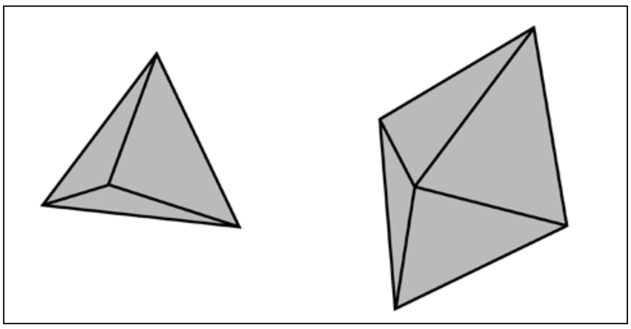

Figure 5: Tetrahedron and triangular bipyramid.

Using ten triangles

There are six possibly distinct polyhedra that can be made with ten equilateral triangles (Ansell *et al,* 1994, Activity 17) and equipped with Polydron tiles children usually discover the whole set quite quickly. Over the course of the workshops each of the polyhedra has acquired a name, but they are not as straightforward as the lower-order deltahedra (Figure 6).

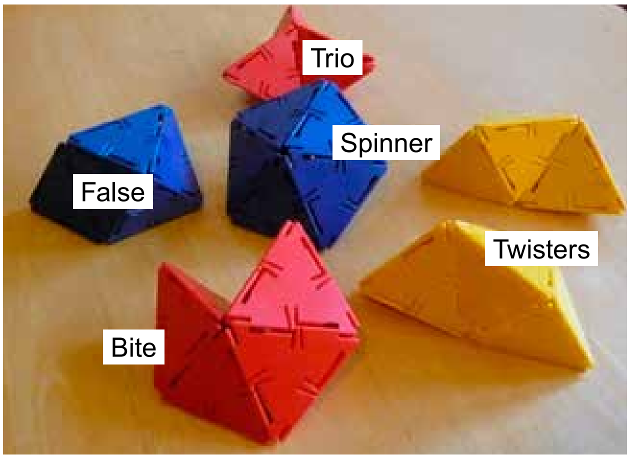

Figure 6: Polyhedra made from 10 equilateral triangles.

The *Spinner*, the *Bite* (which is sometimes called the *Fortune Cookie* or the *Packman*), the *Trio* and the two *Twisters* are all delta-decahedra. But while the first three are distinct, the two *Twisters* are reflections of one another, twisting in different directions. Any net of one *Twister* may be folded inside out to make its reflection, and the children sometimes decide that the two *Twisters* are the same deltahedron.

The sixth, or arguably fifth, polyhedron in the photo can certainly be made from ten equilateral triangles, but none the less it may not have ten triangular faces, so it may not be a delta-decahedron. Three of the pairs of triangles are co-planar: each pair forms a flat, rhombic face. This gives the polyhedron seven faces, four triangles and three rhombuses, so it could be said to be neither a decahedron (with ten faces) nor a deltahedron (with triangle faces). Hence its name, the *False*.

Once we have established our set of polyhedra made with ten triangles, the children can begin to search for their nets. Now we are in completely unexplored territory, as far as I know. I have no idea how many nets there are of, say, the *Twister*, or what they all look like (see Figure 7, over the page). They include several examples of what Adrian Pinel calls "symm-nets", with rotational or reflectional symmetry (Pinel, 2017), although many of the nets have neither.

But what about the 10-triangle ribbon net? Well, just as Evie's 8-triangle ribbon net folded up into an octahedron, so a 10-triangle ribbon net will fold up to into a decahedron. In fact, as Figures 8 to 10, over the page, show, it will fold into more than one.

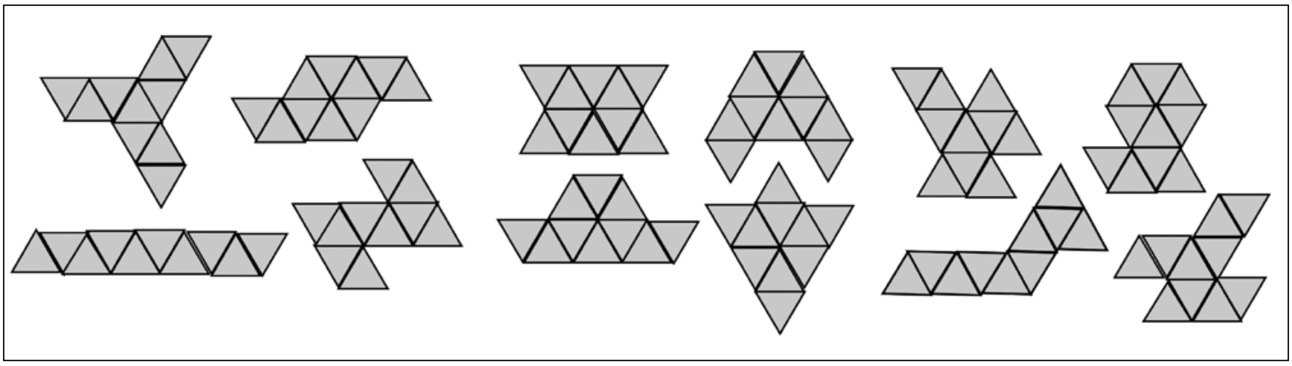

Figure 7: Some examples of *Twister* nets.

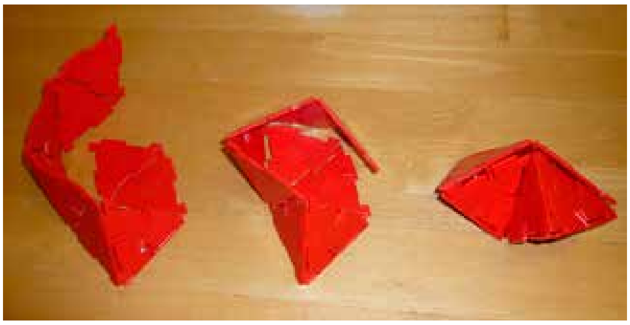

Figure 8: Folding the *Bite* from a ribbon net.

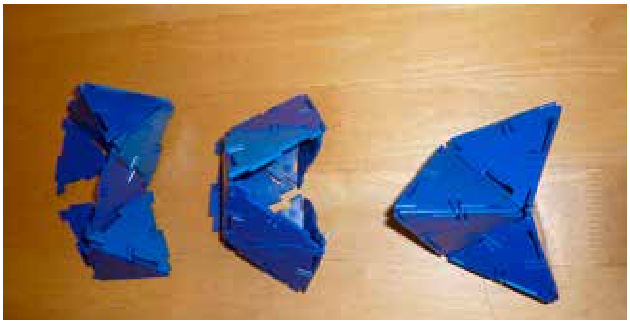

Figure 9: Folding the *Trio* from a ribbon net.

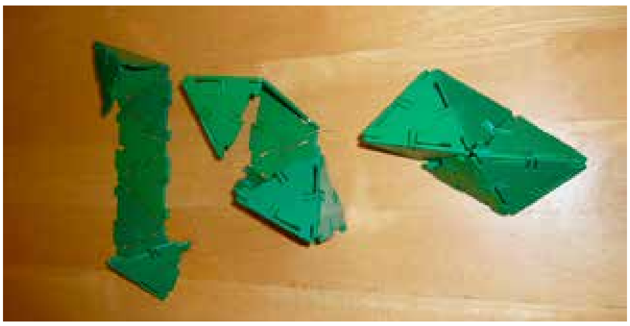

Figure 10: Folding the *Twister* from a ribbon net.

So here we have, not two deltahedra from one net, but three: the *Bite*, the *Trio*, and the *Twister* can all be folded from a ribbon of 10 triangles.

Growing Twisters

Another way to make some, though not all, deltahedra is to join tetrahedra face-to-face. So, for example, a decahedral *Twister* with ten triangular faces can be made by joining four tetrahedra together (see Figure 11).

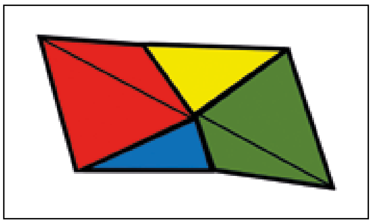

Figure 11: 10-triangle *Twister* made from four tetrahedra.

Another tetrahedron stuck onto the end of the 10-triangle *Twister* forms a 12-triangle *Twister* with five tetrahedra. This too may be folded out of a ribbon net, this time of 12 triangles.

And so, the pattern continues. A *Twister* made with a 14-triangle ribbon net can be made with six tetrahedra; one made with a 16-triangle ribbon net can be made with seven, and so on (see Figure 12, over the page). However long the ribbon net, adding two more triangles will always add another tetrahedron to the *Twister*.

Similarly, working backwards, removing one of the four tetrahedra from the end of the 10-triangle *Twister* produces the *Boat*, composed of three tetrahedra. Removing another one leaves two tetrahedra that form a triangle bipyramid, and removing one of these leaves a single tetrahedron (see Figure 13, over the page). So, although they are too short to show the 'twist', the *Boat*, the triangle bipyramid and the solitary tetrahedron, all of which have ribbon nets, could be regarded as embryonic forms of the *Twister*.

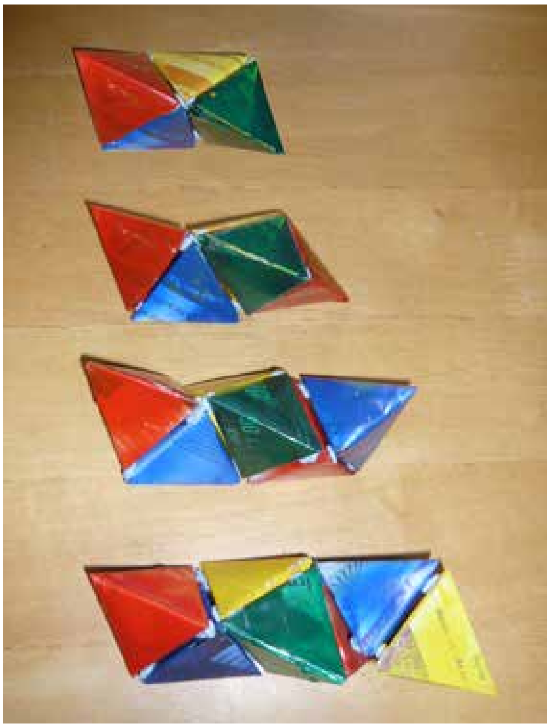

Figure 12: ‘Growing’ the 5-. 6- and 7-tetrahedron *Twisters.*

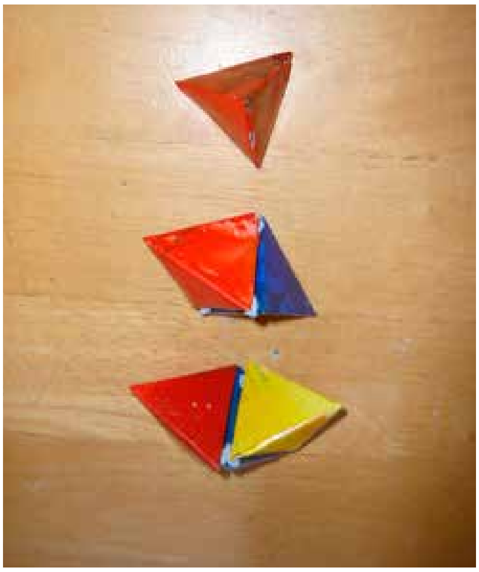

Figure 13: 3-. 2- and 1-tetrahedron *Twisters.*

As we have seen, any net made of a single straight ribbon of triangles can be expanded out and folded into a rigid *Twister*. But it can also be folded up on itself into a pile of connected triangles (Figure 14). This takes up relatively little space, so it might, perhaps, have some practical applications for construction or engineering.

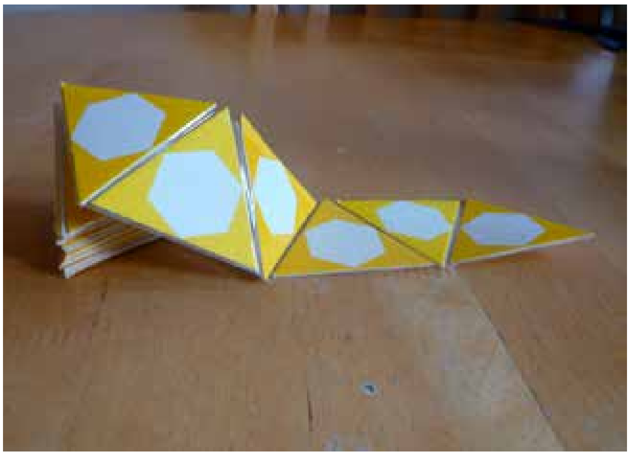

Figure 14: A *Twister* net as a pile of triangles.

Exploration and discovery

In his plenary address to the 2017 ATM Easter Conference, Adrian Pinel asked, “How do we widen access to mathematics for all learners?” He went on to describe how the exploration of tessellations and polyhedra created a community of ‘mathematical thinkers’ in his classroom. Such a community, whether of adults or of children, can be very powerful, and the practical, hands-on investigation of deltahedra offers a rich context to work in. The children’s discoveries in the mathematics enrichment workshops, such as Josh and Len’s ‘double’ net and Evie’s ribbon net, have opened up paths and possibilities that we might otherwise never have noticed, and these are leading to further explorations as we search for and classify new shapes made with triangles. This is what ‘enrichment’ is all about: not the details of the deltahedra and their nets, but the gripping excitement of purposeful exploration and genuine discovery. And that is something that every mathematician, and therefore every child, should experience.

Tandi Clausen-May

References

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Pinel, A. (2017) The role of geometry in fashioning an inclusive mathematics education. *Mathematics Teaching* 259: 7-12.